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On the central limit theorem for series with respect to periodical multiplicative systems. II

S. V. LEVIZOV

Introduction. In [1] we studied the question of the central limit theorem (CLT) for lacunary subsystems of periodical multiplacitve orthonormal systems (PMONS), satisfying the so-called weak lacunarity condition.*) We also defined the concept of the subjection of subsystems to CTL and gave sufficient conditions for being a subjection, too. The sharpness of the given conditions was also shown. The assumption about the boundedness of the sequence $\{p_n\}$ generating the investigated PMONS had an essential importance. The purpose of the present work is to extend these results to the case when $\overline{\lim\limits_{n\to\infty}}\, p_n=+\infty$. We shall also investigate the connection of the rate of the growth of $\{p_n\}$ and the "density" of the lacunary sequence $\{n_k\}$.

1. Sufficient conditions. Let the PMONS $X=\{\chi_n(x)\}$ be defined by means of the sequence $\{p_n\}$. Denote $\{\chi_{n_k}(x)\}$ a lacunary subsystem of X such that the sequence $\{n_k\}$ satisfies the conditions

$$\frac{n_{k+1}}{n_k} \geqq 1+\omega(k) \quad \text{for} \quad k \geqq k_o, \tag{1}$$

where $\omega(k)\downarrow 0$ and $k^{\alpha}\cdot\omega(k)\uparrow\infty$ for some $\alpha, 0<\alpha<1$.

For given sequence $\{n_k\}$ we define, as earlier, λ_k and $\lambda_k^l(q)$ as the quantity of the conjugate pairs and the (l,k)-adjoint numbers (with n_q for fixed q) in the $k-$th block of X, respectively. Also we put $\tilde{p}_k:=\max\{p_i\colon 1\leqq i\leqq k+1\}$; $k=1,2,\ldots$.

Without the restriction $p_n=O(1)$, first we shall give sufficient conditions for the validity of CLT in the case when all of the coefficients α_k are equal to 1.

*) Here and further, in order to avoid the repetitions, for concepts, notations and formulations we refer to [1].

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Theorem C. *Suppose that the sequences* $\{n_k\}$, $\{\omega(k)\}$ *and* $\{p_k\}$ *satisfy condition* (1) *and additionally:*

a)

$$\ln \tilde{p}_k = o\left(\sqrt{f(k+1)} \cdot \omega(f(k+1))\right) \quad as \quad k \to \infty; \tag{2}$$

b) *there exists a real number* η, $0 \leqq \eta \leqq 1$ *such that*

$$\lim_{N \to \infty} \frac{1}{f(N+1)} \sum_{i=1}^{N} \lambda_i = \eta; \tag{3}$$

c) *there exists an absolute constant* $C > 1$ *such that for any fixed* q *and for any* j, $0 \leqq j \leqq k-1$

$$\lambda_k^j(q) \cdot \omega(f(k)) = O(C^{j-k}) \cdot \ln \tilde{p}_k \quad as \quad k \to \infty. \tag{4}$$

Then the subsystem $\{\chi_{n_k}(x)\}$ *is the subject to* CLT.

Proof. The line of the proof follows that of Theorem A in [1] (conditions (2)—(4) of Theorem C are the analogues of conditions (1.2)—(1.4) of Theorem A, respectively). All of the lemmas of § 2 ([1]) remain valid. Lemmas of § 3 need only some light modifications, caused by the estimation of the value $\delta_k = f(k+1) - f(k)$. In our case, using the arguments of [2], we have:

$$\delta_k = O\left\{\frac{\ln p_{k+1}}{\omega(f(k+1))}\right\}.$$

In addition, since $a_k = 1$ for $k = 1, 2, \ldots$, then $b_k = \max\{|a_j| : f(k) < j \leqq f(k+1)\} = 1$ and with condition (2) we conclude that

$$b_k = o\left\{\frac{B_k \cdot \omega(f(k+1))}{\ln \tilde{p}_k}\right\} \quad \text{as} \quad k \to \infty.$$

These facts simplify the proofs of Lemmas 3.2 and 3.3 (see [1]). The completion of Theorem C runs as in § 4 of [1].

Remark 1. It is easy to see that Theorem A is a corollary of Theorem C in the case $p_n = O(1)$ and $a_k = 1$ for $k = 1, 2, \ldots$.

Remarks 2. The sharpness of condition b) of Theorem C follows from the proof of Theorem B ([1]). At the same time the sharpness of conditions a) and c) remains open.

If the coefficients a_k are not all equal we can formulate the following statement, which is a generalization of Theorems A and C.

Theorem D. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$ *any* $\{p_k\}$ *satisfy conditions* (1), (2) *and* (4). *Further let a sequence* $\{a_k\}$ *be such that*

$$A_k^2 = \sum_{i=1}^{k} a_i^2 \to \infty, \quad a_k = O\left(\frac{A_k}{\sqrt{k}}\right) \quad as \quad k \to \infty; \tag{5}$$

and there should exist the limit

$$\eta = \lim_{N\to\infty} \frac{1}{B_N^2} \sum_{j}^{f(N+1)} (a_j \cdot \hat{a}_j) < \infty. \tag{6}$$

Then the subsystem $\{a_k \chi_{n_k}(x)\}$ *is the subject to* CLT (*briefly*: $\{a_k \chi_{n_k}(x)\} \subset \mathrm{CLT}$).

The proof of Theorem D goes analogously to the foregoing reasons.

Remark 3. The additional condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ in this case is essential. Namely, in the course of the proofs of Theorems A and C we used the estimation $|\Delta_k(x)| = o(B_k)$. In Theorems A and C this estimation arises from conditions (1.2) (see [1]) and (2), respectively. But in Theorem D it needs some supplementary calculations. Since the condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ implies that $b_k = O\left(\frac{B_k}{\sqrt{f(k)}}\right)$, hence (2) gives that

$$|\Delta_k(x)| = \sum_{i=f(k)+1}^{f(k+1)} a_i \chi_{n_i}(x) \leqq b_k \cdot \delta_k = O\left(\frac{B_k}{\sqrt{f(k)}}\right) \cdot \frac{\ln p_{k+1}}{\omega(f(k+1))} =$$

$$= o\left(\frac{B_k \cdot \sqrt{f(k+1)}}{\sqrt{f(k)}}\right) = o(B_k) \quad \text{as} \quad k \to \infty,$$

because $f(k+1) \sim f(k)$ $\left(\text{by virtue of } \delta_k < \frac{\ln p_{k+1}}{\omega(f(k+1))} = o(\sqrt{f(k+1)})\right)$. The condition $a_k = O\left(\frac{A_k}{\sqrt{k}}\right)$ holds, e.g., for any non-increasing sequence $\{a_k\}$ $(a_k \geqq 0)$.

Using the methods of proof of the previous theorems it is possible to establish a further analogue of Theorem A.

Theorem E. *Let the sequences* $\{n_k\}$, $\{\omega(k)\}$, $\{a_k\}$ *and* $\{p_k\}$ *satisfy conditions* (1), (4), (6) *and*

$$A_k^2 = \sum_{i=1}^{k} a_i^2 \to \infty, \quad A_k = O(k \cdot a_k); \tag{7}$$

$$b_k = o\left(\frac{B_k \cdot \omega(f(k+1))}{\ln \tilde{p}_k}\right) \quad \text{as} \quad k \to \infty. \tag{8}$$

Then $\{a_k \chi_{n_k}(x)\} \subset CLT$.

Remark 4. The second condition of (7) assures the relation $f(k+1)\sim f(k)$ (in the proofs of Theorems A, C, D this follows form conditions (1.2) and (2), respectively). In the present case we have: $B_k=A_{f(k+1)}=O\big(f(k+1)\cdot a_{f(k+1)}\big)=$ $=O\big(b_k\cdot f(k+1)\big)$, whence (8) gives the required relation. The condition $A_k=$ $=O(k\cdot a_k)$ is fulfilled, e.g., for any non-decreasing sequence $\{a_k\}$.

Concluding the paragraph we note that in Theorems A, C, D, E conditions (1.4) and (4) can be replaced by the next one. Denote by p_k^j the quantity of pairs (n_q, n_r) such that $m_k\le n_q$, $n_r<m_{k+1}$ and $1\le q+r<m_{j+1}$ (j can be equal to 0, 1, ..., $k-1$). Then instead of conditions (1.4) and (4) it is possible to use the condition

$$\sum_{j=0}^{k-1} b_j^2\cdot \varrho_k^j\cdot \delta_j = o(B_k^2) \quad \text{as} \quad k\to\infty. \tag{9}$$

This condition is "cruder" than conditions (1.4) and (4), but, on the other hand, it simplifies the proof of Lemma 3.3 (see [1]) essentially. In this case we immediately obtain the estimation

$$\int_0^1 (\Delta_k^2\cdot \bar{\Delta}_j^2)\,dx \le b_k^2\cdot b_j^2\cdot \varrho_k^j\cdot \delta_j$$

(see the arguments in [1]). Therefore

$$L_N^{(2)} = \sum_{k=1}^{N}\sum_{j=0}^{k-1}\int_0^1 (\Delta_k^2\cdot \bar{\Delta}_j^2)\,dx \le \sum_{k=1}^{N} b_k^2\cdot \sum_{j=0}^{k-1} b_j^2\cdot \varrho_k^j\cdot\delta_j \le$$

$$\le \sum_{k=1}^{N}\Big(\sum_{i=f(k)+1}^{f(k+1)}\Big)\sum_{j=0}^{k-1} b_j^2\cdot\varrho_k^j\cdot\delta_j = {}_N^2 B\, o(B_N^2) = o(B_N^4) \quad \text{as} \quad N\to\infty,$$

that was required. Then we can estimate $L_N^{(3)}$ similarly and the proof of Lemma 3.3 can be completed faster.

The sharpness of condition (9) can be demonstrated by the same counterexamples as the sharpness of (1.4) in Theorem A.

2. How fast can the numbers p_k grow? Comparing Theorem A with Theorems C, D, E we notice that the conditions of type (2) is a "key moment" in the question on CLT for the case $\varlimsup_{n\to\infty} p_n=+\infty$. Now we consider the problem: What kind of growth of $\{p_k\}$ does the fulfilment of condition (2) assure even if $a_k=1$, $\omega(k)=$ $=\frac{1}{k^\alpha}$; $k=1, 2, \ldots$; $\alpha\ge 0$? In this case we have:

$$\ln \tilde{p}_k = o\big(\sqrt{f(k+1)}\cdot (f(k+1))^{-\alpha}\big) = o\big((f(k+1))^{1/2-\alpha}\big). \tag{10}$$

By (10) it follows that when $p_k=O(1)$ then an admissible boundary of the lacunarity is $\alpha=\frac{1}{2}$ (this fact is well known for the Walsh system).

For convenience we assume that the sequence $\{n_k\}$ is "regularly" lacunary, i.e. there exist constants c and d such that $c \geqq d > 0$ and

$$1+\frac{c}{k^\alpha} \geqq \frac{n_{k+1}}{n_k} \geqq 1+\frac{d}{k^\alpha} \tag{11}$$

for some $\alpha > 0$ and $k=1, 2, \ldots$.

In this case we have:

$$m_k \leqq n_{f(k)+1} \leqq n_{f(k)} \cdot \left(1+\frac{c}{(f(k))^\alpha}\right) \leqq \ldots \leqq n_1 \cdot \prod_{i=0}^{f(k)} \left(1+\frac{c}{i^\alpha}\right).$$

Taking the logarithm, we obtain that

$$\ln m_k \leqq \ln n_1 + \sum_{i=1}^{f(k)} \ln \left(1+\frac{c}{i^\alpha}\right). \tag{12}$$

Since $f(k)\uparrow\infty$ and $\ln\left(1+\frac{c}{i^\alpha}\right) = O\left(\frac{1}{i^\alpha}\right)$, then (12) implies that

$$\ln m_k = O\left(\ln n_1 + \sum_{i=1}^{f(k)} \frac{1}{i^\alpha}\right) = O\left(f(k)^{1-\alpha}\right). \tag{13}$$

By (10) and (13) it follows that under assumptions (11) we used relation (13).

$$\ln \tilde{p}_k = \{(\ln m_k)^{(1/2-\alpha)/1-\alpha}\} \tag{14}$$

(here we use the relation $f(k+1) \sim f(k)$).

Since $\ln m_k = \sum_{i=1}^{k} \ln p_i$, thus (14) shows a correlation between the growth of $\{p_k\}$ and the index of the lacunarity. So, if $p_k \sim \exp(k^\beta)$, $\beta > 0$, then $\ln m_k \asymp k^{\beta+1}$ and from (14) we obtain the following sufficient condition

$$\beta < 1-2\alpha. \tag{15}$$

Inequality (15) shows that only an exponential growth of $\{p_k\}$ can force to move away from the boundary $\alpha = \frac{1}{2}$.

Investigating the critical case $\alpha = \frac{1}{2}$ we shall suppose that

$$\frac{n_{k+1}}{n_k} - 1 \asymp \frac{\varphi(k)}{\sqrt{k}},$$

where $\varphi(k)\uparrow\infty$ and $\varphi(k) = o(\sqrt{k})$.

Then for the fulfilment of (2) it is sufficient that

$$\ln \tilde{p}_k = o\left(\varphi(f(k))\right). \tag{16}$$

By simple calculations we obtain the relation

$$\ln m_k = o\big(\sqrt{f(k)} \cdot \varphi(f(k))\big),$$

which implies

$$\ln m_k = o\big(f(k)\big). \tag{17}$$

By (16) and (17) we conclude that if $p_k \sim k^\gamma$ $(\gamma > 0)$, then the following functions $\varphi(k)$ are suitable: $\varphi(k) = (\ln k)^{1+\varepsilon}$ for any $\varepsilon > 0$, $\varphi(k) = \ln k \cdot \ln\ln(k+1)$ and so on.

In particular, if $\{p_k\}$ consists of only the prime numbers (i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ... etc.) then $p_k \sim k \ln k$ and all foregoing statements are valid.

Finally, we remark that under condition (11), by the estimation $\max\limits_{i \le k} \beta_i = O\left(\dfrac{\ln \tilde{p}_k}{\omega(f(k+1))}\right)$, condition (9) can be simplified. In the case $a_k = 1$ $(k = 1, 2, ...)$ it has the form

$$\sum_{j=0}^{k-1} \varrho_k^j = o(k),$$

where we used relaiton (13).

References

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POLYTECHNICAL INSTITUTE, CHAIR OF HIGHER MATHEMATICS